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Moyal Quantization of sdiff($T^{2k/N}$), q-Deformation, and q-Moyal Bracket

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We study the Moyal quantization of the algebras of orthosymplectic superdiffeomorphisms on the supertorus sdiff($T^{2k/N}$). We also examine the connection between the q-deformation and the q-Moyal quantization on sdiff(T^{2k}).

1. INTRODUCTION

The volume-preserving diffeomorphisms algebra sdiff(M) and its N-extended supersymmetric sdiff($M \times G_N$) on the given smooth manifold, arise in diverse areas of theoretical and mathematical physics; for example in the theory of relativistic extended objects termed "p-branes" (Bergshoeff *et al.*, 1988) and its extended supersymmetry. They arise also in the so-called w_∞-supergravity (Sezgin, 1991; Hull, 1991; Bergshoeff *et al.*, 1990b).

Recently much attention has been paid to the quantum version of the above algebraic structures (Floratos, Iliopoulos, and Tiktopoules, 1989; Hoppe and Schaller, 1989; Fairlie, Fletcher and Zachos, 1989; de Witt, Hoppe and Nicolai, 1988; EL Kinani and M. Zakkari, 1995, 1997). In this paper we study a quantum version of these structures by using the $*$-product and the Moyal bracket (MB) and its extension supersymmetry. This work is organized as follows: In Section 2 some general properties of the symplectic diffeomorphism algebra in the $2k$-torus sdiff(T^{2k}) are explained, and the field-theoretic model invariant by sdiff(T^{2k}) is examined. Section 3 is devoted to Moyal quantization of sdiff(T^{2k}); the connection between the q-deformation and the q-Moyal quantization (Dayi, 1996) is pointed out. In Section 4 we study the superalgebras of orthosymplectic superdiffeomorphisms on the supertorus

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$sdiff(T^{2k/N})$ and its quantum version by extending the Moyal formalism. Finally, in the last section we give our conclusion and an outlook for further work.

2. CLASSICAL AND QUANTUM ALGEBRA OF SYMPLECTIC DIFFEOMORPHISMS ON THE TORUS T^{2k}

In this section, we recall the basic notion connected with the algebra of symplectic diffeomorphisms on a smooth manifold T^{2k}, the $2k$-dimensional torus $T^{2k} = S^1 \times S^1 \times \cdots \times S^1$ $(2 - k$ times). The symplectic structure can be represented in term of a canonical constant antisymmetric $2k \times 2k$ matrix which can be chosen as

$$\omega_{ab} = \begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot \\ -1 & 0 & 0 & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 1 \\ \cdot & \cdot & \cdot & -1 & 0 \end{pmatrix}, \qquad a, b = 1, 2, \ldots, 2k \tag{1}$$

Let now $\{\sigma_a\}_{a=1}^{2k}$ denote the corresponding coordinates on T^{2k}, then the infinite-dimensional Lie algebra $sdiff(T^{2k})$ of infinite symplectic diffeomorphisms $\Gamma(\sigma)$ is defined as

$$\begin{aligned} sdiff(T^{2k}) = \Big\{ & \Gamma(\sigma) \in \mathscr{F}un(T^{2k})/[\Gamma_1(\sigma), \Gamma_2(\sigma)] \equiv \{\Gamma_1, \Gamma_2\} \\ & = \omega^{ab} \frac{\partial \Gamma_1}{\partial \sigma_a} \frac{\partial \Gamma_2}{\partial \sigma_b} \Big\} \end{aligned} \tag{2}$$

As seen from equation (2), the Lie commutator is nothing but the canonical Poisson bracket on $\mathscr{F}un(T^{2k})$, where ω^{ab} indicates the inverse matrix of ω_{ab} such that $\omega^{ab}\omega_{cb} = \delta^a_c$. It is natural to choose a basic function $\exp(i\mathbf{m} \cdot \boldsymbol{\sigma})$ where $\mathbf{m} \cdot \boldsymbol{\sigma} = m^j \sigma_j$, with $j = 1, 2, \ldots, 2k$. In terms of this basis function, $sdiff(T^{2k})$ takes the form

$$[\Gamma_{\mathbf{m}}, \Gamma_{\mathbf{n}}] = (\mathbf{m} \wedge \mathbf{n})\Gamma_{\mathbf{m}+\mathbf{n}} \tag{3}$$

where $\mathbf{m} \wedge \mathbf{n} = m^a \omega_{ab} n^b$. Note that the above algebra is invariant by the transformation $\Gamma_{\mathbf{m}} \rightarrow \Gamma_{M\mathbf{m}}$, where M is an element of $sl(2k, \mathbf{Z})$.

For completeness, we consider the central extension of such an algebra, which is given by

$$[\Gamma_{\mathbf{m}}, \Gamma_{\mathbf{n}}] = (m \wedge n)\Gamma_{\mathbf{m}+\mathbf{n}} + A \cdot m \delta_{\mathbf{m}+\mathbf{n},0} \tag{4}$$

where $A = (A_1, A_2, \ldots, A_{2k})$ is the $2k$-tuplet of central charge.

To end this section, we point out that the field theoretic model invariant by sdiff(T^{2k}) is given in (Guendelman *et al.*, 1995) as

$$\mathcal{L} = \frac{1}{4e^2} F^2_{\mu\nu}(\sigma) + \overline{\psi}(i\partial\!\!\!/ - A\!\!\!/(\sigma) - m)\psi \tag{5}$$

where $F_{\mu\nu}(\sigma) = \omega_{ab}\partial_\mu\sigma^a(x)\partial_\nu\sigma^b(x)$ is the antisymmetric tensor field constructed from the canonical symplectic closed two-form $\omega(\sigma) = \omega_{ab}d\sigma^a \times d\sigma^b$, and $\{\sigma_a(x)\}_{a=1}^{2k}$ is the set of $2k$-scalar fields on the ordinary Minkowski space-time, taking values in the symplectic manifold T^{2k}; here $A_\mu = \frac{1}{2}\omega_{ab}$-$\sigma^a(x)\partial_\mu\sigma^b(x)$ is the corresponding vector potential, and ψ denotes the ordinary Dirac fermions. This model is termed "mini QED."

3. QUANTIZATION OF sdiff(T^{2k}), ITS q-DEFORMATION, AND ITS q-MOYAL BRACKET

In this section, we quantize the above algebra; here we adopt the Moyal quantization picture. To do this, we introduce the following star product given by infinite-order differential operators:

$$\begin{aligned}(f * g)(\sigma) &= f(\sigma) \exp\left(i\hbar \frac{\overleftarrow{\partial}}{\partial\sigma_a}\,\omega_{ab}\,\frac{\overrightarrow{\partial}}{\partial\sigma_b}\right) g(\sigma) \\ &= (f(\sigma) \exp(i\hbar\overleftarrow{\partial}_a\omega_{ab}\overrightarrow{\partial}_b) g(\sigma) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (i\hbar)^n \omega_{a_1b_1} \cdots \omega_{a_nb_n}(\partial_{a_1} \cdots \partial_{a_n} f(\sigma))(\partial_{b_1} \cdots \partial_{b_n} g(\sigma))\end{aligned} \tag{6}$$

Hence the Moyal bracket $\{.,.\}_{\mathrm{MB}}$ is defined as

$$\{f, q\}_{\mathrm{MB}} = \frac{1}{i\hbar}(f * g - g * f) \tag{7}$$

Observe here that the classical limit of the star product becomes the ordinary one:

$$(f * g)(\sigma) = f(\sigma)g(\sigma) + O(\hbar^2) \tag{8}$$

and for $\hbar \to 0$, the Moyal bracket approaches the Poisson bracket $\{f, g\}_{\mathrm{MB}} = \{f, g\}_{\mathrm{PB}} + O(\hbar^2)$; the Poisson bracket is $\{f, g\}_{\mathrm{PB}} = \partial_a f(\sigma)\omega_{ab}\partial_b g(\sigma)$ in our notation. The Moyal bracket of the $\Gamma_{\mathbf{m}}$ is easily calculated, and the Moyal quantization of the algebra equation (3)

$$[\Gamma_{\mathbf{m}}, \Gamma_{\mathbf{n}}] = \frac{2}{\hbar} \operatorname{Sin}(\hbar m^a \omega_{ab} n^b) \Gamma_{\mathbf{m}+\mathbf{n}} + A \cdot m \delta_{\mathbf{m}+\mathbf{n},0} \tag{9}$$

At this step we would like to point out that the q-deformation of the above algebra without central terms is obtained in EL Kinani and Zakkari (1997),

$$[\Gamma_{\mathbf{m}}, \Gamma_{\mathbf{n}}]_{(q^{2k(\mathbf{m}\wedge\mathbf{n})}, q^{-2k(\mathbf{m}\wedge\mathbf{n})})} = [m^a \omega_{ab} n^b]_q \Gamma_{\mathbf{m}+\mathbf{n}} \tag{10}$$

where

$$[B, C]_{(q,p)} = qBC - pCB \qquad \text{and} \qquad [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$$

Let us now consider the connection between the q-deformation and the so-called q-Moyal quantization (Dayi, 1996). If we define the q-star product of the $\Gamma_{\mathbf{m}}$ as

$$\Gamma_{\mathbf{m}} *^q \Gamma_{\mathbf{n}} = q^{2k(\mathbf{m}\wedge\mathbf{n})} \Gamma_{\mathbf{m}} * \Gamma_{\mathbf{n}} \tag{11}$$

then our q-Moyal bracket is defined as

$$\{\Gamma_{\mathbf{m}}, \Gamma_{\mathbf{n}}\}_{q-\mathrm{MB}} = \frac{1}{i\hbar} (q^{2k(\mathbf{m}\wedge\mathbf{n})} \Gamma_{\mathbf{m}} * \Gamma_{\mathbf{n}} - q^{-2k(\mathbf{m}\wedge\mathbf{n})} \Gamma_{\mathbf{n}} * \Gamma_{\mathbf{m}}) \tag{12}$$

One can see easily that for $q = e^{-i\hbar/k}$, the q-deformation algebra (10) is nothing but the q-Moyal quantization (12) after an appropriate redefinition of the generators. Hence the q-deformation is equivalent to the q-Moyal quantization obtained from our q-star product (11).

4. SUPERALGEBRA OF ORTHOSYMPLECTIC SUPERDIFFEOMORPHISMS

Let us consider a supertorus $T^{2k/N}$ with local angular coordinates $\{\sigma_a\}_{a=1}^{2k}$ parametrizing the bosonic part T^{2k} and the Grassmann coordinates ξ_j, $j = 1, 2, \ldots, N$ [their Grassmann parities are $P(\sigma_a) = 0$, $P(\xi_j) = 1$]. In the space of superfunction F on the supertorus, one can choose a basis of the form

$$F_{\mathbf{m}, i_1, i_2, \ldots, i_l} = \xi_{i_1} \xi_{i_2} \cdots \xi_{i_l} \exp(i \mathbf{m} \cdot \sigma), \qquad l = 0, 1, \ldots, N \tag{13}$$

On the space F, one can define an orthosymplectic structure by means of the Poisson superbracket

$$\{f, g\}_{\mathrm{PB}} = \omega^{ab} \frac{\partial f}{\partial \sigma_a} \frac{\partial g}{\partial \sigma_b} - (-1)^{P(f)} \sum_{j=1}^{N} \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial \xi_j} \tag{14}$$

where $P(f)$ is the parity of element f, defined as

$$P(F_{\mathbf{m},i_1,i_2,\ldots,i_l}) = \tfrac{1}{2}(1 - (-1)^l) \tag{15}$$

With respect the Poisson superbracket (13), F becomes a Lie superalgebra with the supercommutation relations in the basis (12):

$$\begin{aligned} &\{F_{\mathbf{m},i_1,\ldots,i_l}, F_{\mathbf{n},j_1,\ldots,j_{l'}}]_{\mathrm{PB}} \\ &\quad = (\mathbf{m} \wedge \mathbf{n})F_{\mathbf{m}+\mathbf{n},i_1,\ldots,i_l,j_1,\ldots,j_{l'}} \\ &\qquad - (-1)^{(1-(-1)^l)/2} \sum_{\alpha=1}^{l} \sum_{\beta=1}^{l'} \delta_{i_\alpha,j_\beta} F_{\mathbf{m}+\mathbf{n},i_1,\ldots,\hat{i}_\alpha,\ldots,i_l,j_1,\ldots,\hat{j}_\beta,\ldots,j_{l'}} \end{aligned} \tag{16}$$

The symbols caret means that i_α, etc., is omitted.

Now we return to the central extension of the above superbacket, which is written as

$$\{f, g]_{\mathrm{CPB}} = \{f, g]_{\mathrm{PB}} + A_a C^a(f, g) \tag{17}$$

with nontrivial basic cocycle (see, e.g., Fradkin and Linetsky, 1990), given by the integral

$$C^a(f, g) = \int d\sigma\, f \frac{\partial g}{\partial \sigma_a} \tag{18}$$

where the supervolume element reads

$$d\sigma = \frac{(-1)^{N(N-1)/2}}{(2\pi)^{2k} N!}\, d^{2k}\sigma \varepsilon^{i_1,\ldots,i_N} d\xi_{i_1} \cdots d\xi_{i_N} \tag{19}$$

and the A_a are $2k$ arbitrary supercentral elements with the Grassmann parity $P(A_a) = \frac{1}{2}(1 - (-1)^N) = P_N$. From the cocycle condition of $C^a(f, g)$, namely weak antisymmetric relation, and the Jacobi identities for the Poisson superbracket, the cocycle $C^a(F_{\mathbf{m},i_1,\ldots,i_l}, F_{\mathbf{n},j_1,\ldots,j_l})$ read

$$C^a(F_{\mathbf{m},i_1,\ldots,i_l}, \mathbf{F}_{\mathbf{n},j_1,\ldots,j_{l'}}) = iA_a m^a \delta_{m+n,0} \delta_{l+l',N} \varepsilon_{i_1,\ldots,i_l,j_1,\ldots,j_{l'}} \tag{20}$$

and the algebra (16) becomes

$$\begin{aligned} &\{F_{\mathbf{m},i_1,\ldots,i_l}, F_{\mathbf{n},j_1,\ldots,j_{l'}}]_{\mathrm{PB}} \\ &\quad = (\mathbf{m} \wedge \mathbf{n})F_{\mathbf{m}+\mathbf{n},i_1,\ldots,i_{l'},j_1,\ldots,j_{l'}} \\ &\qquad - (-1)^{(1-(-1)^l/2} \sum_{\alpha=1}^{l} \sum_{\beta=1}^{l'} \delta_{i_\alpha,j_\beta} F_{\mathbf{m}+\mathbf{n},i_1,\ldots,\hat{i}_\alpha,\ldots,i_{l'},j_1,\ldots,\hat{j}_\beta,\ldots,j_{l'}} \\ &\qquad + iA_a m^a \delta_{m+n,0} \delta_{l+l',N} \varepsilon_{i_1,\ldots,i_l,j_1,\ldots,j_{l'}} \end{aligned} \tag{21}$$

Now we quantize the above Poisson superbracket. By extending the Moyal formalism, we define the extended star product as follows:

$$(f * g)(\sigma, \xi) = f(\sigma, \xi) \exp(i\hbar \overleftarrow{\bar{\partial}}_\alpha \omega^{ab} \overline{\bar{\partial}}_b - (-1)^{P(f)} \hbar \sum_{j=1}^{N}$$

$$\times \frac{\overleftarrow{\bar{\partial}}}{\partial \xi_j} \frac{\overline{\bar{\partial}}}{\partial \xi_j} g(\sigma, \xi) \tag{22}$$

The super Moyal bracket is defined as usual by

$$\{f, g]_{\mathrm{MB}} = \frac{1}{i\hbar}(f * g - (-1)^{P(f)P(g)} g * f) \tag{23}$$

After algebraic manipulation, for the symbols corresponding to the basis function (13) we obtain

$$\begin{aligned}
&\{F_{\mathbf{m},i_1,\ldots,i_l}, F_{\mathbf{n},j_1,\ldots,j_{l'}}]_{\mathrm{MB}} \\
&= \frac{1}{i\hbar} \sum_{s=0}^{\min(l,l')} s!\, C_l^s C_{l'}^s (\exp\{i\hbar[-(-1)^{[1-(-1)^l]/2}\hbar](\mathbf{m} \wedge \mathbf{n})\} \\
&\quad - (-1)^{(1-(-1)^l)(1-(-1)^{l'})/4} \exp\{-i\hbar[-(-1)^{[1-(-1)^{l'}]/2}\hbar](\mathbf{m} \wedge \mathbf{n})\} \\
&\quad \times \delta_{i_{l-s+1},j_s} \cdots \delta_{i_l,j_{l'}} F_{\mathbf{m}+\mathbf{n},i_1\ldots i_{l-s}j_{s+1}\ldots j_{l'}}
\end{aligned} \tag{24}$$

where $C_l^s = l!/[s!\,(l-s)!]$. The centerless superalgebra (21) is recovered by setting $\hbar \to 0$.

4. CONCLUSION

In this paper, we have discussed the supervolume-preserving algebra $\mathrm{sdiff}(T^{2k/N})$ and its quantum version by extending the Moyal formalism; we have seen that in this context the quantization means that we replace the infinite-dimensional Lie algebra of $\mathcal{F}\mathrm{un}(T^{2k/N})$ equipped with the Poisson superbracket by the deformed algebra equipped with the Moyal superbracket. We have also examined the connection between the q-deformation and the q-Moyal bracket, which are equivalent modulo normalization. An important issue is to extended the so-called "mini QED" to the supersymmetric case by using the supersymmetric symplectic structure on $\mathrm{sdiff}(T^{2k/N})$, $\omega(\sigma, \xi) = \omega_{ab} d\sigma^a \times d\sigma^b + \frac{1}{2}\Sigma_{j=1}^{N} (d\xi_j)^2$. More analysis on this question will be given elsewhere.

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